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Quantum topology change and large-N gauge theories

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Abstract: We study a model for dynamical localization of topology using ideas from non-commutative geometry and topology in quantum mechanics. We consider a collection X of N one-dimensional manifolds and the corresponding set of boundary conditions (self-adjoint extensions) of the Dirac operator D. The set of boundary conditions encodes the topology and is parameterized by unitary matrices g. A particular geometry is described by a spectral triple $x(g) = (A_X, \mathcal{H}_X, D(g))$. We define a partition function for the sum over all g. In this model topology fluctuates but the dimension is kept fixed. We use the spectral principle to obtain an action for the set of boundary conditions. Together with invariance principles the procedure fixes the partition function for fluctuating topologies. The model has one free-parameter β and it is equivalent to a one plaquette gauge theory. We argue that topology becomes localized at $\beta = \infty$ for any value of N. Moreover, the system undergoes a third-order phase transition at $\beta = 1$ for large-N. We give a topological interpretation of the phase transition by looking how it affects the topology.

Keywords: Models of Quantum Gravity, Non-Commutative Geometry, Matrix Models.

Contents

1. Introduction

A coherent picture embracing both quantum theory and gravity has been a big challenge for over seventy years [1]. The last decades have witnessed a conceptual change on the usual notions of space-time and quantum mechanics. It is generally agreed that at very high energies the conventional ideas about the space-time breaks down, so that the geometrical framework of General Relativity becomes inadequate to describe the non-manifold micro-structure of space-time. In string theory, for instance, X_μ are operators that happen to be interpreted as coordinates of an embedding in a metric space [2]. Many theories also suggest a discrete picture of the space-time at very small distances. Thus, in loop quantum gravity the operators of spatial area and volume have discrete spectra [3]. Furthermore, both string theory and loop quantum gravity strongly indicate a non-commutative structure of the space-time at the Planck scale [3, 4].

Although a complete, non-perturbative theory of quantum gravity is still unknown it is fair to say that some theoretical progress has been achieved. This advance stimulated the development of a growing quantum gravity phenomenology. It is now argued that space-time fluctuations at the scale of quantum gravity may be probed/tested at energies accessible experimentally, now or in a near future. Many experimental proposals rely on departures from the classical Lorentz symmetry due to space-time quantum fluctuations, with different scenarios predicting similar modified dispersion relations [5]. Possible tests include time-of-arrival difference between high-energy photons from gamma-ray bursts and observations of high-energy cosmic rays above the GZK bound [5]. Other interesting experimental set-ups involve analysis of noise in gravity-wave [5] and matter interferometers [6]. There is also some input from current experimental data [7, 8, 9] which can place constraints on the possible scales of space-time fluctuation effects. Finally, cosmological observables such as the cosmic microwave background spectrum may also contain clues to the quantum structure of spacetime [10, 11].

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Any consistent theory of quantum gravity should, in the low energy limit, give us the conventional geometric picture of space-time. In such theory the space-time itself will be dynamically generated. Thus its dimension, the signature of the metric, global topology, causal structure, etc., will be computable (at least in principle) observables of the theory, and not predetermined inputs. However, almost all models assume a given dimensionality, signature and/or topology from the beginning. This is the case in the simplicial approach (quantum Regge calculus [12], dynamical "triangulations" [13], lorentzian triangulations[1] [14], etc.), and in the (perturbative) string scenario where the target space has a fixed dimensionality to be determined later by consistence conditions (e.g. anomaly cancellations). In fact, string calculations assume a fixed background classical space-time. It is hoped that a non-perturbative approach would introduce manifest background independence into the theory, but this remains a conjecture. Loop quantum gravity is already background independent, however specific models are usually set from the start in a 3+1 space-time. There is some progress towards these opens questions in other proposals. For instance, in the causal set theory [16] one starts from a minimum data input, namely a discrete set of events endowed with a causal relation. A poset structure is also shown to arise naturally in the so-called causal spin networks [17].

Many theories suggest that the background of quantum gravity is well described by some type of space-time foam [18, 19], notably spin foam models and related ones. Thus, not only the geometrical properties of the space-time, but its topology is also subject to quantum fluctuations at the quantum gravity scale. There is some phenomenological proposals to uncover possible macroscopic signals of quantum topology fluctuations, see for instance [20]. It is widely believed that topology changes are pure quantum phenomena, and a necessary ingredient for a consistent theory of quantum gravity. For instance, there is no spin-statistics theorem for geons (i.e. soliton-like excitations of a spatial manifold) unless topology changing processes are allowed [21, 22]. In this paper we will study a toy model for topology fluctuations where it will be possible to address dynamical questions in a simpler context.

Since the manifold structure of space-time has to appear at some macroscopic limit, it is natural to expect that one needs a generalization of ordinary geometry, such as non-commutative geometry (NCG), to approach the Planck scale physics. The starting point of NCG [23] is the remarkable observation that one can describe a riemannian manifold $(M, g_{\mu\nu})$ in a purely algebraic way. There is no loss of information if, instead of the data $(M, g_{\mu\nu})$, one is given a triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is the C*-algebra $C^0(M)$ of smooth functions on M, $\mathcal{H}$ is the Hilbert space of L^2-spinors on M, and D is the Dirac operator acting on $\mathcal{H}$. From the Gelfand-Naimark theorem it is known that the topological space M can be reconstructed from the set $\widehat{\mathcal{A}}$ of irreducible representations of $C^0(M)$. Metric is also encoded, and the geodesic distance can be computed from D. In particular one

[1] It is argued that the "effective" dimension of the space-time (associated with the ensemble of random triangulations in the continuum limit) may be different from the dimension of the underlying simplex. For instance, the Hausdorff dimension (d_H) of 2D pure euclidean gravity turns out to be four in the dynamical triangulation approach [15], in contrast with the result $d_H = 2$ in the lorentzian triangulation [14]. However, this is seem somewhat as a "pathology" of the euclidean formulation [14].

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can treat all Hausdorff topological spaces in this way. Given a pair $(M, g_{\mu\nu})$, one can promptly construct the corresponding triple $(C^0(M), L^2(M), D)$. However, not all commutative spectral triples come from a pair $(M, g_{\mu\nu})$. Nevertheless one can always associate a Hausdorff space $M = \widehat{\mathcal{A}}$ to a commutative spectral triple, where $\widehat{\mathcal{A}}$ denotes the set of irreducible representations of $\mathcal{A}$. The space M may not be a manifold and the spectral triple has to be regarded as a generalized geometry.

The framework of NCG suggests a possible approach to quantum gravity based on the spectral principle [24, 25]. Once we trade the original riemannian geometry for its corresponding commutative triple we need a replacement for the Einstein-Hilbert action S_{EH}. According to this principle the action depends only on the spectrum of the Dirac operator D. It is customary to use $\mathrm{Tr}[\chi(D/\Lambda)]$ where χ is a positive cut-off function that goes to zero very fast and Λ is some scale. This action is justified by the fact that when expanded in powers of $1/\Lambda$, the first non constant term is proportional to S_{EH}. The choice of χ does not affect the expansion strongly. The terms are always the same up to constants (couplings) that depend on χ [25]. The so-called spectral action of Chamseddine and Connes [24] corresponds to a simple choice for χ.

Although the spectral principle was initially formulated in terms of riemannian geometry, it can be used in a broader context. Spectral actions can be written for any triple, regardless of whether it comes from a manifold $(M, g_{\mu\nu})$ or not. In the spectral geometry approach it is thus conceivable to write the partition function

$$Z = \sum_{x \in \mathcal{X}} e^{-S[x]}\,, \tag{1.1}$$

where the "sum" is over the set $\mathcal{X}$ of all possible commutative spectral triples and S depends on the spectrum of D. It includes all Hausdorff spaces and therefore all manifolds of all dimensions.

In a previous Letter [26] we introduced a simple discrete model to quantum gravity based on a particular truncation of the sum in (1.1). We discretized (1.1) by sampling the set $\mathcal{X}$ with finite commutative spectral triples $x = (\mathcal{A}, \mathcal{H}, D)$ where the commutative C*-algebra $\mathcal{A}$ has a countable spectrum $\widehat{\mathcal{A}}$. In this approach to discretization there is no need to introduce a lattice or simplicial decomposition of the underlying space. The approximation of $\mathcal{A}$ by a finite dimensional algebra works even if the spectral triple does not come from a manifold. Thus, it gives us a generalization of ordinary discretizations [13, 12, 27]. The model describes the geometry of spaces with a countable number n of points, and is related to the gaussian unitary ensemble of hermitean matrices so that for fixed n the operator D is a $n \times n$ self-adjoint matrix. The average number of points in the universe $\langle n \rangle$, the expectation value $\langle \delta \rangle$ of the dimension, and the metric are macroscopic observables of the theory, obtained after some suitable average (coarse-graining) over the ensemble. We showed that the discrete model has two phases: a finite phase with a finite value of $\langle n \rangle$ and $\langle \delta \rangle = 0$, and an infinite phase with a diverging $\langle n \rangle$ and a finite $\langle \delta \rangle \neq 0$. The critical point was computed as well as the critical exponent of $\langle n \rangle$. Moreover, an upper bound for the order parameter $\langle \delta \rangle$ was found, $\langle \delta \rangle \leq 2$. The discrete model is a pre-geometric one, in the sense that the continuum picture with its geometrical content emerges through a phase transition.

In the present paper we elaborate on another discrete model where the dimension will be kept fixed while the topology fluctuates. Again, we will consider only degrees of freedom associated to pure gravity, i.e. coupling with matter degrees of freedom will not be included. Relying on the framework developed in [28], we consider a collection X of N intervals of length L. For this set of one dimensional manifolds, the momentum operator P plays the role of the Dirac operator. The sum in (1.1) will be over triplets $x = (A_X, \mathcal{H}_X, D = P)$ where A_X is the algebra of continuous functions on X and $\mathcal{H}_X = L^2(X)$. In order to fix the spectral triple, however, we have to consider the self-adjoint extensions of P, i.e. boundary conditions (b.c.). These are labeled by unitary matrices g. Thus, we are lead to compute a partition function over all self-adjoint extensions of P [29]. According to Balachandran et al. [28] the b.c. fixes the global topology of the configuration space. The topology depends on the form of g, and in general is different from the classical one. In particular, it can be a superposition of circles S^1 of different sizes. The definition of the triplets and a short revision of the main arguments in [28] are the subject of section 2. In section 3 we use the spectral principle as a guide to define the analogue of the Einstein-Hilbert action for our system. Together with symmetry requirements, this procedure fixes the partition function. Once we have the partition function for the ensemble of all topologies we are able to study the dynamical localization of topology. This is done in sections 4 and 5, where our main results are discussed. We identify the model (with a parameter β) with the Gross-Witten model which arises from the Wilson's lattice version of YM_2. Namely, the partition function reduces to a generalization of the Dyson's circular unitary ensemble. We numerically verify that the configuration space is a collection of circles of size L, $S^1 \cup S^1 \cup \cdots \cup S^1$, for all finite N at $\beta = \infty$. Topology thus gets localized in this limit. In the large-N limit there is a phase transition at $\beta = \beta_c$. We also give a topological interpretation of the phase transition by looking how it may affects the topology.

2. Fluctuating topology

The connections between topology and quantum mechanics have been clearly exposed in [28]. Here we rephrase the discussion using the language of non-commutative geometry.

Let us consider a collection X of N one dimensional manifolds (intervals) of length L. The corresponding spectral triple will be taken as $x = (A_X, \mathcal{H}_X, D)$ where A_X is the algebra of continuous functions on X and $\mathcal{H}_X = L^2(X)$. The analogue of the Dirac operator D will be the momentum operator P.

Let us consider a simple example where X is a pair of disjoint intervals I_1, I_2. The intervals will be parametrized by a coordinate $x \in [0, L]$. The classical configuration space of a particle living on X is just the union $[0, L] \cup [0, L]$. An element $\psi \in \mathcal{H}_X$ is a pair of functions $\psi_1(x), \psi_2(x)$, $\psi_i : I_i \to \mathbb{C}$ and the scalar product is

$$(\psi, \chi) = \int_0^L dx \sum_{i=1}^{2} (\psi_i^* \chi_i)(x)\,. \tag{2.1}$$

We write the wave-function conveniently as a column vector $\psi(x) = (\psi_1(x), \psi_2(x))^t$ so that the operator $D = P_2$ takes the following matrix form

$$P_2 = \begin{pmatrix} -i\partial_x & 0 \\ 0 & -i\partial_x \end{pmatrix} . \tag{2.2}$$

We have not fixed completely the spectral triple. The operator D is fixed only up to boundary conditions (b.c.) or self-adjoint extensions. Let the eigenfunctions of the operator P_2 be of the form

$$\psi(x) = \begin{pmatrix} A \\ B \end{pmatrix} e^{ipx} , \tag{2.3}$$

where A, B and p are obtained by solving the equation

$$\psi(L) = g\psi(0) , \tag{2.4}$$

with $g \in \mathrm{U}(2)$ parameterizes the b.c. or self-adjoint extensions.

One may ask what geometrical properties of X are determined by such b.c.. The point of view taken by Balachandran et all. in [28] is that a b.c. fixes the global topology. Depending on the form of g, the topology perceived by the quantum particle is quite different from the classical one. Let us look at a couple of examples to clarify this:

$$(a) \qquad g_a = \begin{pmatrix} 0 & e^{i\theta_{12}} \\ e^{i\theta_{21}} & 0 \end{pmatrix} , \tag{2.5}$$

$$(b) \qquad g_b = \begin{pmatrix} e^{i\theta_{11}} & 0 \\ 0 & e^{i\theta_{22}} \end{pmatrix} . \tag{2.6}$$

The probability of finding the particle on the first interval is $\int \psi_1^* \psi_1 dx$, and similarly for the second interval. In the case (a), the density functions $\psi_i^* \chi_i$ satisfy the conditions

$$(\psi_1^* \chi_1)(L) = (\psi_2^* \chi_2)(0) , \tag{2.7}$$

$$(\psi_2^* \chi_2)(L) = (\psi_1^* \chi_1)(0) . \tag{2.8}$$

In other words, the probability densities are the same at the points joined by the thin line (figure 1), and thus the configuration space of the particle is a circle made by joining the two intervals. The eigenfunctions (2.3) are of the form ($A_\pm = \pm \exp\{i(\theta_{12} - \theta_{21})/2\}$)

$$\psi_n^{(+)}(x) = \begin{pmatrix} A_+ \\ 1 \end{pmatrix} e^{i(n + \frac{\theta_{12}+\theta_{21}}{4\pi}) \frac{2\pi x}{L}} , \tag{2.9}$$

$$\psi_n^{(-)}(x) = \begin{pmatrix} A_- \\ 1 \end{pmatrix} e^{i(n + \frac{\theta_{12}+\theta_{21}}{4\pi} + \frac{1}{2}) \frac{2\pi x}{L}} , \tag{2.10}$$

and the spectrum is the set $\{\frac{2\pi}{L}(n + \frac{\theta_{12}+\theta_{21}}{4\pi})\} \cup \{\frac{2\pi}{L}(n + \frac{\theta_{12}+\theta_{21}}{4\pi} + \frac{1}{2})\}$, $n \in \mathbb{Z}$.

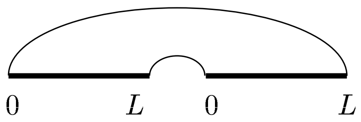

Figure 1: The figure shows a boundary condition corresponding to a circle of size $2L$.

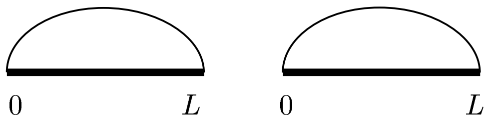

Figure 2: The figure shows a boundary condition corresponding to a pair of circle of sizes L.

In the case (b), the probability densities instead satisfy

$$(\psi_1^*\chi_1)(L) = (\psi_1^*\chi_1)(0), \tag{2.11}$$
$$(\psi_2^*\chi_2)(L) = (\psi_2^*\chi_2)(0) \tag{2.12}$$

Now, as is obvious from figure 2, the underlying configuration space is the union of two circles.

The eigenfunctions are of the form

$$\psi_n^{(u)}(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i(n+\frac{\theta_{11}}{2\pi})\frac{2\pi x}{L}}, \tag{2.13}$$
$$\psi_n^{(d)}(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i(n+\frac{\theta_{22}}{2\pi})\frac{2\pi x}{L}}, \tag{2.14}$$

so that $\mathrm{Spec}(P_2) = \{\frac{2\pi}{L}(n + \frac{\theta_{11}}{2\pi})\} \cup \{\frac{2\pi}{L}(n + \frac{\theta_{22}}{2\pi})\}$.

For most other choices of g, topology is not localized as in these two examples but it is rather a superposition of both, and there is no classical interpretation to it. This happens also for other unitary matrices corresponding to non-trivial b.c..

Notice that the spectrum depends only on the eigenvalues of the matrix g. This is not merely a coincidence for the examples we looked at here. It is easy to see that two matrices g_1 and g_2 that are related by $g_2 = ug_1u^\dagger$ give rise to the same spectrum.

The generalization to arbitrary number N of intervals is straightforward. Our interest is in the operator

$$P_N = \begin{pmatrix} -i\partial_x & 0 & \cdots & 0 \\ 0 & -i\partial_x & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -i\partial_x \end{pmatrix} \tag{2.15}$$

The self-adjoint extensions are labeled by a unitary $N \times N$ matrix g. The topology of the configuration space, as seen by the quantum particle, is dictated by g. Different choices of g can give rise to, for example, a single classical circle or k ($k < N$) disjoint circles. The spectrum of P_N may be written as $\{\frac{2\pi}{L}(n + \frac{\alpha_1}{2\pi})\} \cup \{\frac{2\pi}{L}(n + \frac{\alpha_2}{2\pi})\} \cup \cdots \{\frac{2\pi}{L}(n + \frac{\alpha_N}{2\pi})\}$ where $(e^{i\alpha_1}, e^{i\alpha_2}, \dots, e^{i\alpha_N})$ are the eigenvalues of the matrix g.

The set of all possible matrices g describes the set of topologies that the quantum particle sees. As remarked before, only a small subset have a classical interpretation, since classical topology corresponds to isolated points on the group manifold. Is there any natural sense in which one can associate a probability to a particular topology i.e. for the matrix g from this ensemble? In other words, is it possible to write down some kind of partition function for the g's? A first step towards this direction was done in [28], where a dynamics for the b.c. (quantized boundary conditions) was proposed in the connection picture. Here we follow another route, along the ideas introduced in [26, 29]. Thus, we would like to compute the partition function

$$Z_N = \int [dg]\, e^{-S[x(g)]}\,, \tag{2.16}$$

where $x(g) = (A_X, H_X, P_N(g))$ and $[dg]$ is the $\mathrm{U}(N)$-invariant Haar measure on the group $\mathrm{U}(N)$. Next we look at the probability distribution for the b.c., that is for the unitary matrices g. Then we will be able to ask questions on a possible dynamical localization of topology.

3. Spectral action

In order to compute (2.16) we need to specify a dynamics, i.e. determine an action $S[x]$ for the triple x. Our guide will be the spectral principle introduced in [24]. Their proposal for the action is just the trace of some positive cut-off function of the square of the Dirac operator. In our case, this would imply using the action

$$S_{\mathrm{spec}} = \mathrm{Tr}\, \chi\left(\frac{P_N^2}{\Lambda^2}\right), \tag{3.1}$$

where $\Lambda \equiv 1/L_\Lambda$ is a momentum cut-off. We are not going to adopt S_{spec} as our dynamics but rather use it as a tool to define the analogue of the Einstein-Hilbert action for our system.

In the simplest case $\chi(x)$ is chosen to be 1 for $x < 1$ and smoothly going to zero for $x > 1$. It turns out that in this case S_{spec} is proportional to the number of eigenvalues with absolute value less than Λ, $S_{\mathrm{spec}} \sim Nn(\Lambda)$. Let $\mathcal{P} = (L/2\pi)\, P$ and $\epsilon = (2\pi L_\Lambda/L)^2$. Most of the contribution to the sum in (3.1) comes from modes with $n + \alpha/2\pi$ less than or of order $1/\sqrt{\epsilon} \sim L/L_\Lambda$, whereas higher modes make almost no contribution. Thus, one gets

$$S_{\mathrm{spec}}(\alpha;\epsilon) \sim N \sum_n \chi\left(\epsilon\left(n + \frac{\alpha}{2\pi}\right)\right) \sim N \sum_{|n|}^{1/\sqrt{\epsilon}} 1 \sim N \frac{L}{L_\Lambda}\,. \tag{3.2}$$

As expected, $S_{\mathrm{spec}} \to 0$ for $\epsilon \to \infty$ at fixed N. This is natural since for $L_\Lambda \to \infty$ we are effectively cutting-off all modes.

Another equally valid action that maintains the invariance $\alpha_k \to \alpha_k + 2\pi$ comes from adopting $\chi(P_N^2/\Lambda^2) = e^{-\epsilon \mathcal{P}_N^2}$, giving

$$S_{\mathrm{spec}}(\alpha_i;\epsilon) = \sum_{k=1}^{N} \sum_{n_k=-\infty}^{\infty} e^{-\epsilon(n_k + \frac{\alpha_k}{2\pi})^2}\,. \tag{3.3}$$

We are concerned with the heat-kernel expansion of $S_{\rm spec}$ in (3.3) for $\epsilon \to 0$. This follows at once from the modular transformation,

$$\sum_{n=-\infty}^{\infty} e^{-t(n+z)^2} = \sqrt{\frac{\pi}{t}}\left(1 + 2\sum_{n=1}^{\infty} e^{-\pi^2 n^2/t}\cos(2\pi n z)\right), \tag{3.4}$$

so that the regularized action reads

$$S_{\rm spec}(\alpha;\epsilon) = \sqrt{\frac{\pi}{\epsilon}}\left[N + 2\sum_{k=1}^{N}\sum_{n_k=1}^{\infty} e^{-\pi^2 n_k^2/\epsilon}\cos(n_k\alpha_k)\right], \tag{3.5}$$

and one obtains an expansion for (3.3) in the form

$$S_{\rm spec}(\alpha_i;\epsilon) = a_0(\epsilon) + a_1(\epsilon)\left(\sum_{k=1}^{N}\cos\alpha_k\right) + a_2(\epsilon)\left(\sum_{k=1}^{N}\cos(2\alpha_k)\right) + \cdots . \tag{3.6}$$

Using $\mathrm{Tr}(g+g^\dagger) = 2\sum_k \cos\alpha_k$, $\mathrm{Tr}(g+g^\dagger)^2 = 2N + 2\sum_k \cos 2\alpha_k$, etc. we can re-write the spectral action in terms of the matrix g itself as

$$S_{\rm spec}(g;\epsilon) = b_0(\epsilon) + b_1(\epsilon)\,\mathrm{Tr}(g+g^\dagger) + b_2(\epsilon)\,\mathrm{Tr}(g+g^\dagger)^2 + \cdot\cdot . \tag{3.7}$$

We are now in position to define the dynamics of the model. As explained in section 3, the degrees of freedom are given by the boundary conditions determined by unitary matrices g. The dynamics will be defined by

$$S_N(g,\beta) = \frac{N\beta}{2}\,\mathrm{Tr}(g+g^\dagger), \tag{3.8}$$

where the factor $N/2$ is for later convenience. This choice can be motivated as follows. When the spectral principle is used for spectral triples coming from riemannian geometry, the first non-trivial term in the expansion of $S_{\rm spec}$ is the Einstein-Hilbert action S_{EH} up to a constant that depends on the particular choice of χ. This allow us to identify the second term in (3.7) as the analogue of S_{EH} for the system at hand. This is nothing but Wilson's action for a 2-d Yang-Mills gauge theory on a single plaquette [30, 31].

We would like to point out that the inclusion of higher order terms in (3.8) give us models related to string theory considered in [32].

4. Localization of topology

As we noted earlier, the spectrum of the operator P_N is unchanged when the matrix g is conjugated by a unitary matrix u. One can see that the action (3.8) and the corresponding partition function (2.16) are both invariant under $g \to ugu^\dagger$.

In terms of the eigenvalues $e^{i\alpha_j}$ of the matrix g, the partition function becomes [33]

$$Z_N(\beta) = \int_0^{2\pi} [d\alpha_j]\Delta(\{\alpha_i\})\bar{\Delta}(\{\alpha_i\})e^{-S_N(\beta)}, \tag{4.1}$$

where $\Delta(\{\alpha_i\})$ is the Vandermonde determinant

$$\Delta(\{\alpha_i\}) = \prod_{i<j}(e^{i\alpha_i} - e^{i\alpha_j})\,, \tag{4.2}$$

the normalization is $Z_N(\beta = 0) = 1$ and

$$[d\alpha_j] \equiv \frac{1}{N!(2\pi)^N}\prod_{j=1}^{N} d\alpha_j\,. \tag{4.3}$$

The action (3.8) has been extensively studied in the literature in connection with YM_2, and has interesting properties in the large N limit [31]. For finite N, using the identity [35]

$$\frac{1}{N!}\int_0^{2\pi}\prod_{k=1}^{N} d\alpha_k \prod_{\ell=1}^{N} g(\alpha_\ell)\prod_{i<j}|e^{i\alpha_i} - e^{i\alpha_j}|^2 = \det\left[\int_0^{2\pi} d\alpha g(\alpha)e^{i\alpha(i-j)}\right]_{i,j=1,\ldots,N}\,, \tag{4.4}$$

with $g(\alpha) = e^{-N\beta\cos\alpha}$, it is easy to show that

$$Z_N(\beta) = \det\left[I_{|i-j|}(N\beta)\right]_{i,j=1,\ldots,N}\,, \tag{4.5}$$

where $I_\nu(x)$ is the modified Bessel function of the first kind.

Let us sum up the argument developed up to now. We are considering a collection of N disjoint compact 1D manifolds. A point x in this collection is a union of circles and intervals with some b.c., corresponding to a particular self-adjoint extension of the momentum operator. Accordingly, to each element x we assign a given topology, parameterized by a unitary matrix $g \in \mathrm{U}(N)$ (that is, a boundary condition). Since in our model Z_N is the partition function for the set of all self-adjoint extensions of the momentum operator, i.e. all points x, it is sensible to interpret

$$P_N(x) = \frac{e^{-S[x]}}{Z_N}\,, \tag{4.6}$$

as the probability weight of having a configuration space with the topology of x. It is clear from section 2 that the topology will be in general "fuzzy", i.e., it may not admit a classical interpretation. Regarding model (4.1) we would like to ask whether or not topology gets localized around a classical configuration for some value of β. In other words, is it possible that $P_N(x,\beta)$ gets localized around some particular manifold x?

We stress that the topology of x, or the boundary condition (2.4) is determined by g, but the probability measure in (4.8),

$$P_N(g,\beta) = \frac{1}{Z_N(\beta)}\frac{e^{-\mathcal{H}_N(\alpha_i,\beta)}}{N!(2\pi)^N}\,, \tag{4.7}$$

depends only on the eigenvalues $e^{i\alpha_k}$ of g. It does not picks out a topology but rather an orbit of g under conjugation. In other words, there is not a one-to-one correspondence between the set of all topologies and the eigenvalues of g, as mentioned in section 2. The only exception is the identity matrix $\mathrm{Spec}(g) = \{1, 1, \ldots, 1\}$, which corresponds to the disjoint union $S^1 \cup S^1 \cup S^1 \cdots \cup S^1$ of circles of size L. Thus only in this case one may speak of a dynamical localization of topology.

We would like to remark that Z_N may be interpreted as the partition function (at fixed temperature T) of a 1D plasma of equal charged point-particles constrained to move in a thin circular wire of radius one immersed in a 2D world (plane). Notice that the partition function reads

$$Z_N(\beta) = \int [d\alpha_k] e^{-\mathcal{H}_N(\alpha_i,\beta)}, \tag{4.8}$$

where

$$\mathcal{H}_N = N\beta \sum_{k=1}^{N} \cos\alpha_k - 2\sum_{i<j} \ln|e^{i\alpha_i} - e^{i\alpha_j}|. \tag{4.9}$$

The second-term in the "hamiltonian" $\mathcal{H}_N$ is the 2-body repulsive Coulomb potential, whereas the first term represents a periodic potential with strength given by $N\beta$. It is well-known that at $\beta = 0$ the system displays only a single phase over all the temperature scale, characterized by a long-range order of crystalline type [34]. The spectral density $\sigma_N(\alpha) = \left\langle \sum_k \delta(\alpha - \alpha_k)\right\rangle_N / N$ is uniform around the unity circle, $\sigma_N(\alpha, \beta \to 0) = 1/2\pi$, and the topology is "fuzzy".

However, it is conceivable that such situation does not hold at finite β. Thus, at some value β_c the strength of the periodic potential may be enough to disorder the crystal structure, leading to a melting of the Dyson crystal into a new phase. This conjecture is supported by a numerical analysis of some "observables". In particular, by means of (4.5) it is possible to shown numerically that

$$\langle \cos\alpha_\ell \rangle_N(\beta) = -\frac{1}{N^2}\frac{\partial}{\partial\beta} \ln Z_N(\beta) = \begin{cases} 0 \text{ if } \beta \to 0\,, \\ 1 \text{ if } \beta \to \infty\,. \end{cases} \tag{4.10}$$

For $\beta \to 0$ the eigenvalues become uniformly distributed around the unity circle, the topology is fuzzy, and $\langle \cos\alpha_\ell \rangle_N \to 0$ as expected. On the other hand, for $\beta \to \infty$ the periodic potential overcome the level repulsion and the eigenvalues tend to concentrate around the origin, i.e. matrices $g \sim \mathbb{I}$ are favored. This is a clear signature of a dynamical localization of topology at $\beta = \infty$ for any value of N.

5. Topology and the third order phase transition

Gross and Witten have shown that the one-plaquette model described by (3.8) and (4.8) undergoes a third order phase transition at $\beta = 1$ in the large-N limit [31]. In this section we would like to discuss whether this phase transition has any consequences to classical topology in our model of fluctuating topologies. One has to keep in mind that topology is described by the matrix g of boundary conditions. However, the dynamics depends only on the eigenvalues of g. In other words, a single set of eigenvalues determines a submanifold of boundary conditions. Since the model is not very sensitive to the topology, we do not expect strong topological changes as we tune β across the critical point. The only point where topology is sharply affected is for $\beta \to 0$ as explained in the last section. Nevertheless, it is possible to give a topological interpretation for the phase transition.

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Let us summarize the results we need from [31]. For large values of β, the density of eigenvalues $\sigma(\alpha,\beta)$ is strongly peaked near $\alpha = 0$, whereas the density is almost uniform over the unit circle for $\beta \simeq 0$. More precisely, each phase is characterized by an appropriate spectral density (we change the domain of α from $[0, 2\pi]$ to $[-\pi, \pi]$)

$$\sigma(\alpha,\beta) = \frac{1}{2\pi}(1+\beta\cos\alpha)\,, \qquad -\pi \le \alpha \le \pi\,; \tag{5.1}$$

$$\sigma(\alpha,\beta) = \frac{\beta}{\pi}\cos\frac{\alpha}{2}\sqrt{\beta^{-1}-\sin^2\frac{\alpha}{2}}\,, \qquad -\alpha_c \le \alpha \le \alpha_c\,, \qquad \sin^2\frac{\alpha_c}{2} = \beta^{-1}\,, \tag{5.2}$$

valid for $\beta \le 1$ and $\beta \ge 1$, respectively. The signature of the phase transition at $\beta_c = 1$ is clear. For $\beta \gg 1$ the spectral density has support at a small region around $\alpha \simeq 0$, and the probability of finding an eigenvalue outside of this region is zero. As we decrease β the support of $\sigma_N(\alpha,\beta)$ becomes a larger arc of the unity circle around $\alpha = 0$. For $\beta > \beta_c$ there always be a gap (forbidden region in the eigenvalues space) on the unity circle around $\alpha = \pi$. The gap is closed at $\beta = \beta_c$.

Consider a matrix g of the form

$$g = \begin{pmatrix} B(k) & 0 \\ 0 & S(N-k) \end{pmatrix}, \tag{5.3}$$

where S is an arbitrary unitary and $B(k)$ is the $k \times k$ matrix

$$B(k) = \begin{pmatrix} 0 & w_1 & \cdots & 0 \\ 0 & 0 & w_2 & \vdots \\ \vdots & \ddots & \ddots & w_{k-1} \\ w_k & \dots & 0 & 0 \end{pmatrix} \tag{5.4}$$

with $w_i = e^{i\alpha_i}$. Thus, this b.c. has a big circle of size $k\,L$ as a classical manifold. The big circle is made of intervals number $1, 2, 3, \dots, k$. The remaining $(N-k)$ intervals are connected by an arbitrary boundary condition given by S and may not admit a classical interpretation. Furthermore, any other matrix that is related to (5.4) by a permutation also has a classical big circle of size $k\,L$.

Let us call $C_k \subset \mathrm{U}(N)$ the subset of boundary conditions of type (5.3) and its permutations.

The k eigenvalues of $B(k)$ is a subset of $\mathrm{Spec}(g)$. To find them we write

$$[B(k)]_{ij} = \omega_i\,\delta^j_{(i+1)\mathrm{mod}(k)}\,. \tag{5.5}$$

It follows that $B(k)^k = e^{i\gamma}\,\mathbb{I}$, where $e^{i\gamma} = \omega_1\omega_2\cdots\omega_k$. Therefore the eigenvalues are $\lambda_m = \exp(i\frac{2\pi m+\gamma}{k})$, $m \in \{0, 1, \dots, k-1\}$. The eigenvector corresponding to λ_m is $v_m = (1, \frac{\lambda_m}{w_1}, \frac{\lambda_m^2}{w_1 w_2}, \dots, \frac{\lambda_m^n}{w_1 w_2 \dots w_n}, \dots, \frac{w_k}{\lambda_m})$. Notice that the eigenvalues of $B(k)$ are equally spaced. They occur at the vertices of a regular polygon inscribed in the unity circle.

The existence of a gap for $\beta > \beta_c$ means that the probability density has a support on a subset $H(\beta) \subset \mathrm{U}(N)$ of all boundary conditions.

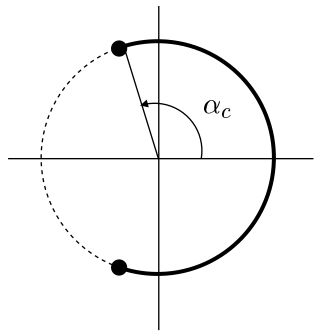

Figure 3: The support of the spectral density is represented on the circle as a dark line interval determined by the angle α_c.

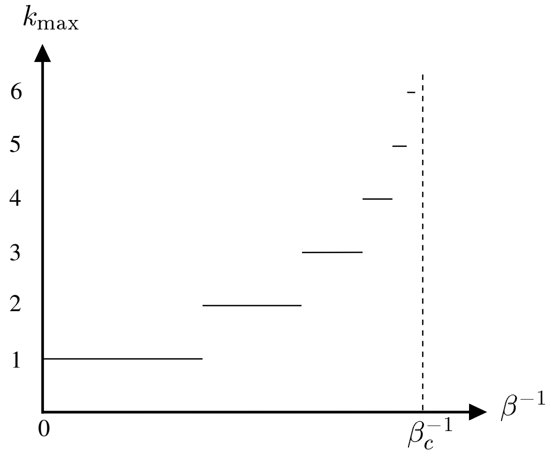

Figure 4: Plot of $k_{\max}$ as a function of β^{-1}.

Let us consider figure 3. The dark region is the support of $\sigma_N(\alpha, \beta^{-1})$ and $\alpha_c = \alpha_c(\beta^{-1})$ is an increasing function of β^{-1}. For $\beta^{-1} \to 0$ we have $\alpha_c \approx 0$. In other words, the largest classical big circle has size $k = L$. Let us see what we need to have a classical big circle of size $k = 2\,L$ inside the allowed region. The eigenvalues of $B(2)$ are $\exp\left(i\frac{\gamma}{2}\right)$ and $\exp\left(i\frac{\gamma}{2} + i\pi\right)$. Suppose we have $\alpha_c = \frac{\pi}{2} + \epsilon$. Many classical circles of size $2\,L$ given by $B(2)$ will be suppressed. However the b.c. such that $\frac{\pi}{2} - \epsilon < \gamma < \frac{\pi}{2} + \epsilon$ will be allowed. For smaller values of α_c, $k = 1$ is the biggest classical circle possible. A similar argument shows that classical big circles with size $k = m\,L$ will only show up for

$$\alpha_c \geq \frac{m-1}{m}\pi\,. \tag{5.6}$$

In other words, for a fixed β, if we look at the intersection of H and C_k we see that H excludes all sets C_k for k larger than some value $k_{\max}$. In figure 4 we plot $k_{\max}$ as a function of β^{-1}. The graph looks like a staircase function. However as we approach β_c, the variable $k_{\max}$ tends to ∞ (in the thermodynamic limit $N \to \infty$) and at the same time the sizes of the plateaus go to zero.

We can also look at another quantity: $\ell(\beta) = \frac{k_{\max}}{N}$. In the limit $N \to \infty$,

$$\ell(\beta) = \begin{cases} 0 \text{ if } \beta \geq \beta_c\,, \\ 1 \text{ if } \beta < \beta_c\,. \end{cases} \tag{5.7}$$

Thus we have the following qualitative picture of the phase transition at β_c. Above β_c we only find classical circles of finite size. There are other b.c. but all classical circles are at most $k_{\max} L$ in size. Below β_c there are classical circles of arbitrary sizes. The order parameter is just $\ell(\beta)$.

6. Conclusions

The above model and the one in [26, 29] are not intended to be realistic. In particular, time does not appear at all in our present model. One can imagine that an equilibrium configuration has been attained, in which the fluctuations in the dimension, metric and signature of the space-time had already been partially localized. Our model would thus be an effective theory describing the topology fluctuations, controlled by the parameter β. This is conceptually similar to the approach followed in [22]. Rather our intention here is to furnish a hint on how NCG, in the formulation embodied in (1.1), may be used to tackle some difficult, open questions in quantum gravity. Thus, in this paper we asked if dynamics can fix topology somehow. These simple models capture some main features of more elaborate ones, and may lead to further insights. This is in accordance with current views on universality in quantum gravity [13, 36]. The idea of universality is enforced here by the connection with random matrix theory. Thus, it can be shown that the upper bound for the dimension observable $\langle\delta\rangle$ found in [26] does not change if instead of complex self-adjoint matrices D one considers real self-adjoint matrices, corresponding to the gaussian orthogonal ensemble [37]. Moreover, the key role played by the eigenvalues of the Dirac operator in GR and in the spectral action approach was emphasized in [25]. They are diffeomorphism-invariant functions of the metric and can be taken as the dynamical variables of GR. In our model they are also the natural dynamical variables due to the connection with random matrix theory.

Inspired by ideas from topology in quantum mechanics and relying on the framework of the non-commutative geometry we have set-up a simple model to study fluctuations in topology for a collection of N one-dimensional manifolds. The model has one free-parameter β and its partition function reduces to the partition function of the one plaquette $\mathrm{U}(N)$ gauge theory. Although our simple dynamics is not particularly sensitive to the topology of the underlying configuration space (since it depends only on the eigenvalues of the unitary matrices g parameterizing the boundary conditions), we have argued that topology gets localized at $\beta \to \infty$ for any value of N. For large-N the model has a third-order phase transition at $\beta_c = 1$. Topology is not, in general, localized for $\beta > \beta_c$ and large-N, however some topologies are excluded due to the finite support of the spectral density in this range of β. Thus it seems possible that, in more realistic models, topology can be indeed fixed by the dynamics.

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The model discussed in the present work points to the remark that an eventual theory of quantum gravity at the Planck scale may possible contain more degrees of freedom than what one would naively expect based on the macroscopic space-time physics [28]. There are many possible ways to extend this and the related work [26]. Notice that the present work and [26] are somehow complementary. Whereas in the latter we have studied fluctuations in the dimension, here we have focused on topology fluctuations keeping the manifold dimension fixed. Thus, it would be interesting to workout a model including both types of fluctuations, with degrees of freedom associated with topology and geometrical dynamics. Another possibility is to include couplings with matter degrees of freedom. We hope that the toy models discussed here in connection with the partition function (1.1) set the stage towards its evaluation in a more realistic scenario where a phenomenological approach can be eventually pursued.

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